9

THE SZEKERES CLASS OF VACUUM SOLUTIONS

The first exact solution to be published which describes a collision between plane waves was in fact that of Szekeres (1970). This describes the collision of plane gravitational waves with step wavefronts. The solution discussed in Chapter 3, which describes the collision between impulsive gravitational waves, was published a little later by Khan and Penrose (1971). Following this, a substantial paper in which the subject is analysed in detail was produced by Szekeres (1972). This paper includes a derivation of the field equations in the form given in Chapter 6, a general class of exact solutions which includes the two mentioned above as special cases, and a discussion of the singularities that arise. The purpose of this chapter is to discuss the properties of this general class of solutions.

9.1 The solution in region IV

The vacuum field equations appropriate to region IV of the colliding plane wave problem may be taken here in the form of equations (6.22*a*–*f*). These are second order differential equations for the four metric functions $U(u,v)$, $V(u,v)$, $W(u,v)$ and $M(u,v)$.

Equation (6.22*a*) may immediately be integrated as in (6.24) to give

$$e^{-U} = f + g \tag{9.1}$$

where $f = f(u)$ and $g = g(v)$ are arbitrary decreasing functions which, according to the appropriate boundary conditions (7.3), are required to satisfy $f(0) = \frac{1}{2}$, $f'(0) = 0$ and $g(0) = \frac{1}{2}$, $g'(0) = 0$.

In the Szekeres class of solutions, the approaching waves have constant aligned polarization. In these solutions therefore

$$W = 0, \tag{9.2}$$

and the main equations (6.22*d*, *e*) reduce to the single equation

$$2V_{uv} = U_u V_v + U_v V_u. \tag{9.3}$$

For this, Szekeres has obtained the solution

$$V = -2k_1 \tanh^{-1}\left(\frac{\frac{1}{2}-f}{\frac{1}{2}+g}\right)^{1/2} - 2k_2 \tanh^{-1}\left(\frac{\frac{1}{2}-g}{\frac{1}{2}+f}\right)^{1/2} \tag{9.4}$$

which contains two arbitrary constants[1] k_1 and k_2. The expression (9.4) may also be written in the alternative form

$$e^V = \left(\frac{\sqrt{\frac{1}{2}+g}-\sqrt{\frac{1}{2}-f}}{\sqrt{\frac{1}{2}+g}+\sqrt{\frac{1}{2}-f}}\right)^{k_1} \left(\frac{\sqrt{\frac{1}{2}+f}-\sqrt{\frac{1}{2}-g}}{\sqrt{\frac{1}{2}+f}+\sqrt{\frac{1}{2}-g}}\right)^{k_2}. \tag{9.5}$$

With this expression for V, the remaining equations in (6.22) may be integrated to give

$$\begin{aligned} M = &-\log(cf'g') - \tfrac{(k_1^2+k_2^2+2k_1k_2-1)}{2}\log(f+g) \\ &+\tfrac{k_1^2}{2}\log(\tfrac{1}{2}-f)+\tfrac{k_2^2}{2}\log(\tfrac{1}{2}+f)+\tfrac{k_2^2}{2}\log(\tfrac{1}{2}-g)+\tfrac{k_1^2}{2}\log(\tfrac{1}{2}+g) \\ &+2k_1k_2\log\left(\sqrt{\tfrac{1}{2}-f}\sqrt{\tfrac{1}{2}-g}+\sqrt{\tfrac{1}{2}+f}\sqrt{\tfrac{1}{2}+g}\right) \end{aligned} \tag{9.6}$$

where c is a constant.

As anticipated in the discussion in Section 7.2, it may be observed that this expression contains the necessary multiples of $\log(\frac{1}{2}-f)$ and $\log(\frac{1}{2}-g)$ that are required to cancel the effects of the unbounded term $\log f'g'$ on the boundary. If the leading terms in the expansions for f and g are

$$f = \tfrac{1}{2} - (c_1u)^{n_1} + \dots, \qquad g = \tfrac{1}{2} - (c_2v)^{n_2} + \dots \tag{9.7}$$

where $n_i \geq 2$, $i = 1, 2$, then e^{-M} is continuous across the boundaries if

$$k_1{}^2 = 2(1-1/n_1), \qquad k_2{}^2 = 2(1-1/n_2). \tag{9.8}$$

It may thus be observed that the constants k_1 and k_2 are restricted to the range satisfying

$$1 \leq k_i{}^2 < 2, \qquad i = 1, 2. \tag{9.9}$$

---

[1] It may be noticed that the constants used here are minus one half of those used by Szekeres.

It is also appropriate to choose

$$c = (c_1 n_1 c_2 n_2)^{-1} \tag{9.10}$$

to achieve the usual flat metric (3.6) in region I.

When evaluating the components of the Weyl tensor, it is convenient to introduce the new function

$$H = 2k_1\sqrt{\tfrac{1}{2}-f}\sqrt{\tfrac{1}{2}+g} - 2k_2\sqrt{\tfrac{1}{2}-g}\sqrt{\tfrac{1}{2}+f}. \tag{9.11}$$

With this, it can be seen that

$$V_u = -\frac{f'}{f+g}H_f, \qquad V_v = \frac{g'}{f+g}H_g, \tag{9.12}$$

and the non-zero Weyl tensor components are

$$\begin{aligned}
\Psi_0^\circ &= -\frac{g'^2}{2(f+g)}H_{gg} - \frac{g'^2}{4(f+g)^2}H_g(1-H_g^2)\\
\Psi_2^\circ &= -\frac{f'g'}{4(f+g)^2}(1-H_fH_g)\\
\Psi_4^\circ &= \frac{f'^2}{2(f+g)}H_{ff} + \frac{f'^2}{4(f+g)^2}H_f(1-H_f^2).
\end{aligned} \tag{9.13}$$

In his paper, Szekeres chooses

$$f = \tfrac{1}{2} - (c_1 u)^{n_1} \qquad \text{and} \qquad g = \tfrac{1}{2} - (c_2 v)^{n_2} \tag{9.14}$$

exactly. In this case, the terms $\frac{k_1^2}{2}\log(\frac{1}{2}-f)$ and $\frac{k_2^2}{2}\log(\frac{1}{2}-g)$ in (9.6) exactly cancel the term $\log(cf'g')$, and M is given by

$$e^{-M} = \frac{(f+g)^{(k_1^2+k_2^2+2k_1k_2-1)/2}}{(\frac{1}{2}+f)^{k_2^2/2}(\frac{1}{2}+g)^{k_1^2/2}\Big(\sqrt{\frac{1}{2}-f}\sqrt{\frac{1}{2}-g}+\sqrt{\frac{1}{2}+f}\sqrt{\frac{1}{2}+g}\Big)^{2k_1k_2}}. \tag{9.15}$$

This form is still completely general since it has, in effect, simply used the coordinate freedom (6.7). However, it is sometimes convenient to retain this freedom, so this restriction will not be made in the following section.

It may now be pointed out that, with the restriction (9.14), the above solution includes the Khan–Penrose (1971) solution for colliding impulsive waves when $n_1 = n_2 = 2$, $k_1 = k_2 = 1$. It also includes the Szekeres (1970) solution for colliding step waves when $n_1 = n_2 = 4$, $k_1 = k_2 = \sqrt{3/2}$.

9.2 The approaching waves

Having obtained an exact solution in region IV, the question now is to find the initial conditions which give rise to it. In fact, it is quite simple to use the method described in Section 7.2 to extend any solution in region IV back into regions II, III and I. We obtain the corresponding solutions in regions II and III simply by replacing g by $\frac{1}{2}$ and f by $\frac{1}{2}$ alternately. In this section we will concentrate on region II, and put $U = -\log\sqrt{\frac{1}{2}+f}$. The equivalent solution in region III can be obtained from this by replacing f by g, and by interchanging u and v.

Using the suggested method, it can be seen that the solution in region II must have the line element

$$\mathrm{d}s^2 = 2e^{-M}\mathrm{d}u\mathrm{d}v - (\tfrac{1}{2}+f)\left(e^V\mathrm{d}x^2 + e^{-V}\mathrm{d}y^2\right) \tag{9.16}$$

where, retaining the coordinate freedom in u,

$$\begin{aligned} f &= \tfrac{1}{2} - (c_1u)^{n_1} + \ldots \\ e^V &= \left(\frac{1-\sqrt{\frac{1}{2}-f}}{1+\sqrt{\frac{1}{2}-f}}\right)^{k_1}, \qquad \frac{k_1^2}{2} = 1 - \frac{1}{n_1} \\ e^{-M} &= -\frac{f'(\frac{1}{2}+f)^{(k_1^2-1)/2}}{c_1n_1(\frac{1}{2}-f)^{k_1^2/2}}. \end{aligned} \tag{9.17}$$

In order to interpret this solution as a plane wave, it is appropriate to transform the metric to the Brinkmann form

$$\mathrm{d}s^2 = 2\mathrm{d}\tilde{u}\mathrm{d}r + h(\tilde{u})(\tilde{x}^2 - \tilde{y}^2)\mathrm{d}\tilde{u}^2 - \mathrm{d}\tilde{x}^2 - \mathrm{d}\tilde{y}^2 \tag{9.18}$$

by putting

$$\begin{aligned} \tilde{x} &= \sqrt{\tfrac{1}{2}+f}\; e^{V/2}x \\ \tilde{y} &= \sqrt{\tfrac{1}{2}+f}\; e^{-V/2}x \\ r &= u - \tfrac{1}{4}(U_u - V_u)e^M\tilde{x}^2 - \tfrac{1}{4}(U_u + V_u)e^M\tilde{y}^2 \\ \tilde{u} &= -\int \frac{(\frac{1}{2}+f)^{(k_1^2-1)/2}}{c_1n_1(\frac{1}{2}-f)^{k_1^2/2}}\frac{\mathrm{d}f}{\mathrm{d}u}\mathrm{d}u. \end{aligned} \tag{9.19}$$

The wave profile in equation (9.18) now takes either of the forms

$$h(\tilde{u}) = c_1\delta(\tilde{u}) \qquad \text{if} \quad n_1 = 2, \quad k_1 = 1 \tag{9.20}$$

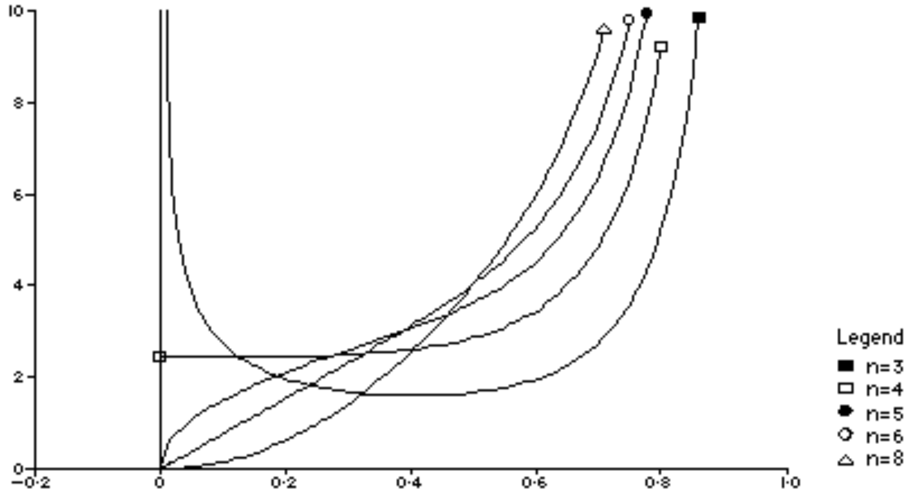

Figure 9.1 Some wave profiles for the approaching waves in the Szekeres solution with $f = 1/2 - u^n$. The wavefront is unbounded if $2 < n < 4$. It contains a step if $n = 4$, and is smooth if $n > 6$. All profiles become unbounded as u approaches 1.

or[2]

$$h(\tilde{u}) = c_1^2 n_1^2 \frac{k_1}{4}\left(1 - \frac{2}{n_1}\right)\frac{(\frac{1}{2} - f)^{1/2-2/n_1}}{(\frac{1}{2} + f)^{3-2/n_1}}\Theta(\tilde{u}) \qquad \text{if} \quad n_1 > 2. \tag{9.21}$$

Particular cases in which $c_1 = 1$ and $f = \frac{1}{2} - u^{n_1}$ are illustrated in Figure 9.1 for $n_1 = 3, 4, 5, 6$ and 8.

It may be recalled from (4.5) that $\Psi_4 = h(\tilde{u})$. Thus, it can be clearly seen that the approaching gravitational wave in region II is an impulsive wave if $n_1 = 2$, has an unbounded wavefront if $2 < n_1 < 4$, has a step wavefront if $n_1 = 4$, has a continuous wavefront if $n_1 > 4$, and has a smooth wavefront if $n_1 > 6$. It may also be observed that, if $n_1 \geq 4$, Ψ_4 is monotonically increasing.

9.3 The singularity structure

In this family of solutions, there is always a scalar polynomial curvature singularity in region IV on the surface on which $f + g = 0$. This can be seen by computing the scalar invariant given here by

$$I = 2e^{2M}(3\Psi_2^{\circ 2} + \Psi_0^\circ \Psi_4^\circ). \tag{9.22}$$

---

[2] It may be noticed that an expression for the profile given by Halil (1979) appears to be incorrect.

Using (9.6) and (9.13), I can be seen to be unbounded when $f+g=0$ for all values of k_1 and k_2 in the required range (9.9). This is similar to the singularity in region IV of the Khan–Penrose solution which, of course, it includes as a special case.

Another singularity occurs in cases in which a wave with an unbounded wave front moves in a region where the opposing wave is nonzero. For example, if $n_2 \neq 2$ and $2 < n_1 < 4$ so that $1 < k_1^2 < \frac{3}{2}$, then I is unbounded on the boundary $u=0$ between regions III and IV. Similarly, the boundary $v=0$, $u \geq 0$ is singular if $n_1 > 2$ and $2 < n_2 < 4$ so that $1 < k_2 < \frac{3}{2}$. Apart from cases containing impulsive waves, where $n_1 = 2$ or $n_2 = 2$, the boundaries of region IV are therefore only regular if $n_i \geq 4$ for $i = 1, 2$, which implies the further restriction $\frac{3}{2} \leq k_i^2 < 2$ on (9.9). These singularities, however, are distribution-valued and their physical significance requires further investigation.

The coordinate singularities in regions II and III, however, are not so obviously similar to the fold singularities of the Khan–Penrose solution. The Szekeres solutions are only flat behind the wave front in the special case when $n_1 = n_2 = 2$. In the general case, regions II and III are curved and the analysis given in Section 8.2 cannot be applied.

However, we may notice that in all cases with $n_1 > 2$, Ψ_4 becomes unbounded as $f \to -1/2$ in region II. This clearly indicates that, in these cases, the singularity in this region is a non-scalar curvature singularity. This has been confirmed in rigorous calculations by Konkowski and Helliwell (1989). It has also been argued in the previous chapter that these singularities have similar properties to the fold singularities that occur for colliding impulsive waves as described in Section 8.2 in terms of the behaviour of neighbouring geodesics.

A similar non-scalar curvature singularity also occurs in region III if $n_2 > 2$, as Ψ_0 then becomes unbounded as $g \to -1/2$.